# Boundedness of Pseudo-Differential Operators on $L^p$, Sobolev, and Modulation Spaces

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**Abstract.** We introduce new classes of modulation spaces over phase space. By means of the Kohn-Nirenberg correspondence, these spaces induce norms on pseudo-differential operators that bound their operator norms on $L^p$–spaces, Sobolev spaces, and modulation spaces.

**Key words:** pseudo-differential operators, modulation spaces, Sobolev spaces, short–time Fourier transforms,
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## 1. Introduction

Pseudo-differential operators are discussed in various areas of mathematics and mathematical physics, for example, in partial differential equations, time-frequency analysis, and quantum mechanics [19, 18, 21, 32, 34]. They are defined as follows.

Let $\sigma$ be a tempered distribution on phase space $\mathbb{R}^{2d}$, that is, $\sigma \in \mathcal{S}'(\mathbb{R}^{2d})$ where $\mathcal{S}(\mathbb{R}^{2d})$ denotes the space of Schwartz class functions. The pseudo-differential operator $T_\sigma$ corresponding to the symbol $\sigma$ is given by

$$T_\sigma f(x) = \int \sigma(x,\xi)\, \widehat{f}(\xi)\, e^{2\pi i x\cdot\xi}\, d\xi, \quad f \in \mathcal{S}(\mathbb{R}^d).$$

Here, $\widehat{f}$ denotes the Fourier transform of $f$, namely,

$$\widehat{f}(\xi) = \mathcal{F}f(\xi) = \int f(x)\, e^{-2\pi i x\cdot\xi}\, dx.$$

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One of the central goals in the study of pseudo-differential operators is to obtain necessary and sufficient conditions for pseudo-differential operators to extend boundedly to function spaces such as $L^p(\mathbb{R}^d)$ [3, 5, 20, 33]. A classical result in this direction is the following.

For $m \in \mathbb{R}$, we let $S^m$ consist of all functions $\sigma$ in $C^\infty(\mathbb{R}^d \times \mathbb{R}^d)$ such that for any multi-index $(\alpha, \beta)$, there is $C_{\alpha,\beta} > 0$ with

$$\left|\left(\partial_x^\beta \partial_\xi^\alpha \sigma\right)(x, \xi)\right| \leq C_{\alpha,\beta}(1 + |\xi|)^{m-|\alpha|}.$$

For $\sigma \in S^0(\mathbb{R}^d)$, it is known that $T_\sigma$ acts boundedly on $L^p(\mathbb{R}^d)$, $p \in (1, \infty)$. A consequence of this result is that if $\sigma \in S^m$, then $T_\sigma$ is a bounded operator mapping $H^p_{s+m}(\mathbb{R}^d)$ to $H^p_s(\mathbb{R}^d)$, where $H^p_s(\mathbb{R}^d)$ is the Sobolev Spaces of order $s \in \mathbb{R}$; for more details see Wong's book [32]. Similarly, in [33], Wong obtains weighted $L^p$–boundedness results for pseudo-differential operators with symbols in $S^m$.

Smoothness and boundedness of symbols though are far from being necessary (nor sufficient) for the $L^p$-boundedness of pseudo-differential operators. In fact, every symbol $\sigma \in L^2(\mathbb{R}^{2d})$ defines a so-called Hilbert–Schmidt operator and Hilbert-Schmidt operators are bounded, in fact, compact operators on $L^2(\mathbb{R}^d)$. Non-smooth and unbounded symbols have been considered systematically in the framework of modulation spaces, an approach that we continue in this paper.

Modulation spaces were first introduced by Feichtinger in [9] and they have been further developed by him and Gröchenig in [8, 9, 12, 10, 11, 13]. In the following, set $\phi(x) = e^{-\pi\|x\|^2/2}$ and let the dual pair bracket $(\cdot, \cdot)$ be linear in the first argument and antilinear in the second argument.

**Definition 1** (Modulation spaces over Euclidean space)**.** *Let $M_\nu$ denote modulation by $\nu \in \mathbb{R}^d$, namely, $M_\nu f(x) = e^{2\pi i t\cdot\nu} f(x)$, and let $T_t$ be translation by $t \in \mathbb{R}^d$, that is, $T_t f(x) = g(x - t)$.*

*The short-time Fourier transform $V_\phi f$ of $f \in \mathcal{S}'(\mathbb{R}^d)$ with respect to the Gaussian window $\phi$ is given by*

$$V_\phi f(t, \nu) = \mathcal{F}\big(f\, T_t \phi\big)(\nu) = (f, M_\nu T_t \phi) = \int f(x)\, e^{-2\pi i x \nu} \phi(x - t)\, dx\,.$$

*The modulation space $M^{pq}(\mathbb{R}^d)$, $1 \leq p, q \leq \infty$, is a Banach space consisting of those $f \in \mathcal{S}'(\mathbb{R}^d)$ with*

$$\|f\|_{M^{pq}} = \|V_\phi f\|_{L^{pq}} = \Big(\int \Big(\int |V_\phi f(t, \nu)|^p\, dt\Big)^{1/p} d\nu\Big)^{1/q} < \infty\,,$$

*with usual adjustment of the mixed norm space if $p = \infty$ and/or $q = \infty$.*

Roughly speaking, distributions in $M^{pq}(\mathbb{R}^d)$ 'decay' at infinity like a function in $L^p(\mathbb{R}^d)$ and have the same local regularity as a function whose Fourier transform is in $L^q(\mathbb{R}^d)$.

The boundedness of pseudo-differential operators on modulation spaces are studied for various classes of symbols, for example, in [5, 7, 15, 16, 27, 28, 30, 31]. In [27, 28] for example, Toft discusses boundedness of pseudo-differential operators on weighted modulation spaces. In [5], Nicola and Cordero describe a class of pseudo-differential operators with symbols $\sigma$ in modulation spaces for which $T_\sigma$ is bounded on $L^p(\mathbb{R}^d)$.

The modulation space membership criteria on Kohn–Nirenberg symbols used in [5, 7, 27, 28] do not allow to require different decay in $x$ and $\xi$ of $\sigma(x,\xi)$. In the recently developed sampling theory for operators, though, a separate treatment of the decay of $x$ and $\xi$ was beneficial [17, 23, 24]. In fact, this allows to realize canonical symbol norms of convolution and multiplication operators as modulation space norms on Kohn-Nirenberg symbols. Motivated by this work, we give the following definition.

**Definition 2** (Modulation spaces over phase space)**.** *The symplectic Fourier transform of* $F \in \mathcal{S}(\mathbb{R}^{2d})$ *is given by*

$$\widetilde{\mathcal{F}}F(t,\nu) = \int_{\mathbb{R}^{2d}} e^{-2\pi i[(x,\xi),(t,\nu)]} F(x,\xi)\,dx\,d\xi, \tag{1.1}$$

*where* $[(x,\xi),(t,\nu)]$ *is the symplectic form of* $(x,\xi)$ *and* $(t,\nu)$ *defined by* $[(x,\xi),(t,\nu)] = x\cdot\nu - \xi\cdot t$. *Analogously, symplectic modulation* $\widetilde{M}_{(t,\nu)}$ *is* $\widetilde{M}_{(t,\nu)}F(x,\xi) = e^{2\pi i[(x,\xi),(t,\nu)]}F(x,\xi)$.

*The symplectic short-time Fourier transform* $\widetilde{V}_\phi f$ *of* $F \in \mathcal{S}'(\mathbb{R}^d)$ *is given by*

$$\begin{aligned} \widetilde{V}_\phi F(x,t,\xi,\nu) &= \widetilde{\mathcal{F}}\big(F\,T_{(x,\xi)}\phi\big)(t,\nu) = (F, \widetilde{M}_{(\nu,t)}T_{(x,\xi)}\phi) \\ &= \iint e^{-2\pi i(\widetilde{x}\nu - \widetilde{\xi}t)} F(\widetilde{x},\widetilde{\xi})\varphi(\widetilde{x}-x,\widetilde{\xi}-\xi)\,d\widetilde{x}\,d\widetilde{\xi}. \end{aligned} \tag{1.2}$$

*The modulation space over phase space* $\widetilde{M}^{p_1p_2q_1q_2}(\mathbb{R}^{2d})$, $1 \le p_1,p_2,q_1,q_2, \le \infty$, *is the Banach space consisting of those* $F \in \mathcal{S}'(\mathbb{R}^{2d})$ *with*

$$\begin{aligned} &\|F\|_{\widetilde{M}^{p_1p_2q_1q_2}} = \|\widetilde{V}_\phi F\|_{L^{p_1p_2q_1q_2}} \\ &= \Big(\int\Big(\int\Big(\int\Big(\int |(\widetilde{V}_\psi F)(x,t,\xi,\nu)|^{p_1}\,dx\Big)^{p_2/p_1}dt\Big)^{q_1/p_2}d\xi\Big)^{q_2/q_1}d\nu\Big)^{1/q_1} \\ &< \infty, \end{aligned} \tag{1.3}$$

*with usual adjustments if* $p_1 = \infty$, $p_2 = \infty$, $q_1 = \infty$, *and/or* $q_2 = \infty$.

Note that the order of the list of variables in (1.2) is crucial as it indicates the order of integration in (1.3). We choose to list first the time variable $x$ followed by the time-shift variable $t$. The time variables are followed by the frequency variable $\xi$ and the frequency-shift variable $\nu$. Alternative orders of integration were considered, for example, in [2, 5, 27, 28].

Below, $\mathcal{L}(X,Y)$ denotes the space of all bounded linear operators mapping the Banach space $X$ to the Banach space $Y$; $\mathcal{L}(X,Y)$ is equipped with the operator norm. Below, the conjugate exponent of $p \in [1,\infty]$ is denoted by $p'$. Our main result follows.

**Theorem 3.** *Let* $p_1,p_2,p_3,p_4,q_1,q_2,q_3,q_4 \in [1,\infty]$. *Then there exists* $C > 0$ *such that*

$$\|T_\sigma\|_{\mathcal{L}(M^{p_1q_1},M^{p_2q_2})} \le C\,\|\sigma\|_{\widetilde{M}^{p_3p_4q_3q_4}}, \quad \sigma \in \widetilde{M}^{p_3p_4q_3q_4}(\mathbb{R}^{2d}), \tag{1.4}$$

*if and only if*

$$\frac{1}{p_1'} + \frac{1}{p_2} \le \frac{1}{p_3} + \frac{1}{p_4}, \qquad p_4 \le \min\{p_1', p_2\}, \tag{1.5}$$

$$\frac{1}{q_1'} + \frac{1}{q_2} \le \frac{1}{q_3} + \frac{1}{q_4}, \qquad q_4 \le \min\{q_1', q_2\}. \tag{1.6}$$

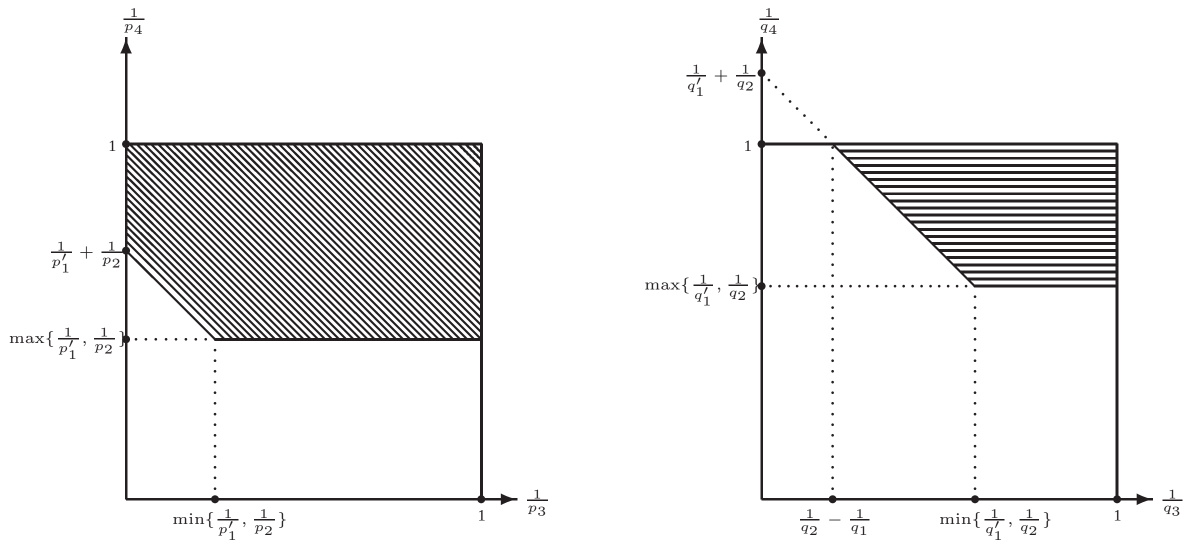


Figure 1: For fixed $p_1, p_2$ and $q_1, q_2$, we mark the regions of $(\frac{1}{p_3}, \frac{1}{p_4})$ and $(\frac{1}{q_3}, \frac{1}{q_4})$ for which every $\sigma \in \widetilde{M}^{p_3 p_4 q_3 q_4}(\mathbb{R}^{2d})$ induces a bounded operator $T_\sigma : M^{p_1 q_1}(\mathbb{R}^d) \to M^{p_2 q_2}(\mathbb{R}^d)$. In fact, for $(\frac{1}{p_3}, \frac{1}{p_4})$ and $(\frac{1}{q_3}, \frac{1}{q_4})$ in the hashed region, there exists $C > 0$ with $\|T_\sigma\|_{\mathcal{L}(M^{p_1 q_1}, M^{p_2 q_2})} \leq C \, \|\sigma\|_{\widetilde{M}^{p_3 p_4 q_3 q_4}}$. The conditions on the time decay parameters $p_1, p_2, p_3, p_4$ are independent of the conditions on the frequency decay parameters $q_1, q_2, q_3, q_4$.

Theorem 12 below is a variant of Theorem 3 that involves symbols in weighted modulation spaces.

Observe that (1.5) depends only on the parameters $p_i$, while (1.6) depends analogously only on the parameters $q_i$. That is, the conditions on decay in time and on decay in frequency, or, equivalently, on smoothness in frequency and on smoothness in time, on the Kohn-Nirenberg symbol are linked to the respective conditions on domain and range of the operator, but time and frequency remain independent of one another. See Figure 1 for an illustration of conditions (1.5) and (1.6).

An $L^p$–boundedness result for the introduced classes of pseudo-differential operators follows.

**Corollary 4.** *Let* $p, p_3, p_4, q, q_3, q_4 \in [1, \infty]$. *Assume*

$$\frac{1}{p'} + \frac{1}{q} \leq \frac{1}{p_3} + \frac{1}{p_4}, \quad p_4 \leq \min\{p', q\},$$

*and*

$$\begin{cases} \frac{1}{p} + \frac{1}{q} & \leq \frac{1}{q_3} + \frac{1}{q_4}, \quad q_4 \leq \min\{p, q\}, \quad \textit{if} \quad p, q \in [1, 2], \\ \frac{1}{p} + \frac{1}{q'} & \leq \frac{1}{q_3} + \frac{1}{q_4}, \quad q_4 \leq \min\{p, q'\}, \quad \textit{if} \quad 1 \leq p \leq 2 \leq q, \\ \frac{1}{p'} + \frac{1}{q'} & \leq \frac{1}{q_3} + \frac{1}{q_4}, \quad q_4 \leq \min\{p', q'\}, \quad \textit{if} \quad 2 \leq \min\{p, q\}, \\ \frac{1}{p'} + \frac{1}{q} & \leq \frac{1}{q_3} + \frac{1}{q_4}, \quad q_4 \leq \min\{p', q\}, \quad \textit{if} \quad 1 \leq q \leq 2 \leq p. \end{cases}$$

*Then* $T_\sigma : L^p(\mathbb{R}^d) \to L^q(\mathbb{R}^d)$ *is bounded and there exists a constant* $C > 0$ *such that*

$$\|T_\sigma\|_{\mathcal{L}(L^p, L^q)} \leq C \, \|\sigma\|_{\widetilde{M}^{p_3 p_4 q_3 q_4}}, \quad \sigma \in \widetilde{M}^{p_3 p_4 q_3 q_4}(\mathbb{R}^{2d}).$$

Corollary 4 encompasses, for example, the space of Hilbert–Schmidt operators on $L^2(\mathbb{R}^d)$, namely

$$HS\big(L^2(\mathbb{R}^d)\big) = \big\{T_\sigma : \sigma \in \widetilde{M}^{2,2,2,2}(\mathbb{R}^{2d}) = L^2(\mathbb{R}^{2d})\big\} \subset \mathcal{L}(L^2(\mathbb{R}^d), L^2(\mathbb{R}^d)).$$

Moreover, Corollary 4 reconfirms also $L^2$–boundedness of Sjöstrand class operators [25, 26],

$$Sj \subset \{T_\sigma : \sigma \in \widetilde{M}^{\infty,1,\infty,1}(\mathbb{R}^{2d})\} \subset \mathcal{L}\Big(L^2(\mathbb{R}^d), L^2(\mathbb{R}^d)\Big).$$

Using the weighted version of Theorem 3, namely, Theorem 12, we get the following boundedness result for Sobolev spaces.

**Corollary 5.** *Let $p_1, p_2, p_3, p_4 \in [1,\infty]$ and $s \in \mathbb{R}$. Let $w$ be a moderate weight function on $\mathbb{R}^{4d}$ satisfying*

$$w(x,t,\nu,\xi) \leq \big(1+|\xi|^2\big)^{s/2}\big(1+|\nu+\xi|^2\big)^{s/2}, \quad x,t,\nu,\xi \in \mathbb{R}^d.$$

*Assume that*

$$\frac{1}{p_1'} + \frac{1}{p_2} \leq \frac{1}{p_3} + \frac{1}{p_4}, \quad p_4 \leq \min\{p_1', p_2\}.$$

*Then*

$$\|T_\sigma\|_{\mathcal{L}(H_s^{p_1}, H_s^{p_2})} \leq C\,\|\sigma\|_{\widetilde{M}_w^{p_3,p_4,1,1}}, \quad \sigma \in \widetilde{M}_w^{p_3,p_4,1,1}(\mathbb{R}^{2d}),$$

*for some constant $C > 0$.*

The paper is structured as follows. Section 2 discusses mixed norm spaces and modulation spaces over Euclidean and over phase space in some detail. In Section 3, our boundedness results for pseudo-differential operators with symbols in modulation spaces over phase space are compared to results in the literature. Finally, in Section 4 we prove our main results, Theorem 3, Corollary 4, and Theorem 12.

## 2. Background on modulation spaces

In the following, $x, \xi, t, \nu$ denote $d$-dimensional Euclidean variables. If not indicated differently, integration is with respect to the Lebesgue measure on $\mathbb{R}^d$.

Let $r = (r_1, r_2, \ldots, r_n)$ where $1 \leq r_i < \infty$, $i = 1, 2, \ldots, n$. The mixed norm space $L^r(\mathbb{R}^n)$ is the set of all measurable functions $f$ on $\mathbb{R}^n$ for which

$$\|F\|_{L^r} = \Big(\int_{\mathbb{R}} \cdots \Big(\int_{\mathbb{R}} \Big(\int_{\mathbb{R}} |F(x_1,\ldots,x_n)|^{r_1}\, dx_1\Big)^{r_1/r_2} dx_2 \ldots\Big)^{r_n/r_{n-1}} dx_n\Big)^{1/r_n}$$

is finite [1]. $L^r(\mathbb{R}^n)$ is a Banach space with norm $\|\cdot\|_{L^r}$. Similarly, we define $L^r(\mathbb{R}^n)$ where $r_i = \infty$ for some indices $i$.

If $n = 2d$, $r_1 = r_2 = \cdots = r_d = p$ and $r_{d+1} = \cdots = r_{2d} = q$, then we denote $L^r(\mathbb{R}^{2d})$ by $L^{pq}(\mathbb{R}^{2d})$. Similarly, if $n = 4d$ and $r_1 = r_2 = \cdots = r_d = p_1$, $r_{d+1} = \cdots = r_{2d} = p_2$, $r_{2d+1} = \cdots = r_{3d} = p_3$ and $r_{3d+1} = \cdots = r_{4d} = p_4$, we write $L^{p_1p_2p_3p_4}(\mathbb{R}^{4d}) = L^r(\mathbb{R}^{4d})$.

Let $w$ be a nonnegative measurable function on $\mathbb{R}^n$. We define $L^r_w(\mathbb{R}^n)$ to be the space all $f$ on $\mathbb{R}^n$ for which $wf$ is in $L^r(\mathbb{R}^n)$. $L^r_w(\mathbb{R}^n)$ is a Banach space with norm given by

$$\|f\|_{L^r_w} = \|wf\|_{L^r}.$$

In time-frequency analysis, it is advantageous to consider moderate weight functions $w$. To define these, let $\mathbb{R}^+_0$ be the set of all nonnegative points in $\mathbb{R}$. Any locally integrable function $v : \mathbb{R}^n \to \mathbb{R}^+_0$ with

$$v(x+y) \leq v(x)v(y)$$

is called submultiplicative. Moreover, if $w : \mathbb{R}^n \to \mathbb{R}^+_0$ is locally integrable with

$$w(x+y) \leq Cw(x)v(y),$$

$C > 0$, and $v$ submultiplicative, then $w$ is called moderate.

The short-time Fourier transform of a tempered distribution $f \in \mathcal{S}'(\mathbb{R}^n)$ with respect to the window $\psi \in \mathcal{S}(\mathbb{R}^n)$ is given by

$$V_\psi f(x,\xi) = \mathcal{F}(fT_x\overline{\psi})(\xi) = (f, M_\xi T_x \psi)$$

where $M_\xi$ and $T_x$ denote modulation and translation as defined above.

With $\phi(x) = e^{-\pi\|x\|^2/2}$, $w$ moderate on $\mathbb{R}^{2d}$, and $p, q \in [1,\infty]$, the modulation space $M^{pq}_w(\mathbb{R}^d)$ is the set of all tempered distributions $f \in \mathcal{S}'(\mathbb{R}^d)$ such that

$$V_\phi f \in L^{pq}_w(\mathbb{R}^{2d}).$$

with respective Banach space norm. Clearly, if $w \equiv 1$, then $M^{pq}_w(\mathbb{R}^d) = M^{pq}(\mathbb{R}^d)$. Moreover, for any $s \in \mathbb{R}$ let

$$w_s(x,\xi) = \left(1 + |\xi|^2\right)^{s/2}$$

and denote $M^{pq}_{w_s}(\mathbb{R}^d)$ by $M^{pq}_s(\mathbb{R}^d)$.

Note that replacing the Gaussian function $\phi$ in the definition of modulation spaces by any other $\psi \in \mathcal{S}(\mathbb{R}^d)\setminus\{0\}$ defines the same space and an equivalent norm, a fact that will be used extensively below.

Recall that the Sobolev space $H^p_s(\mathbb{R}^d)$ consist of all tempered distributions $u \in \mathcal{S}'(\mathbb{R}^d)$ for which $\|u\|_{H^p_s} = \|T_{w_s}u\|_{L^p} < \infty$ [27]. For any $s \in \mathbb{R}$ and $1 \leq q \leq p \leq r \leq q' \leq \infty$ we have

$$M^{pq}_s(\mathbb{R}^d) \subseteq H^r_s(\mathbb{R}^d), \tag{2.1}$$

and for some $C > 0$,

$$\|f\|_{H^r_s} \leq C\|f\|_{M^{pq}_s}, \quad f \in M^{pq}_s(\mathbb{R}^d).$$

Similarly, $1 \leq q' \leq r \leq p \leq q \leq \infty$ implies

$$H^r_s(\mathbb{R}^d) \subseteq M^{pq}_s(\mathbb{R}^d), \tag{2.2}$$

and for some constant $C > 0$,

$$\|f\|_{M_s^{pq}} \leq C\|f\|_{H_s^r}, \quad f \in H_s^r(\mathbb{R}^d).$$

Let $\mathcal{F}L^p(\mathbb{R}^d)$ be the space of all tempered distributions $f$ in $\mathcal{S}'(\mathbb{R}^d)$ for which there exists a function $h \in L^p(\mathbb{R}^d)$ such that $\hat{h} = f$. Then $\mathcal{F}L^p(\mathbb{R}^d)$ is a Banach space equipped with the norm

$$\|f\|_{\mathcal{F}L^p} = \|h\|_{L^p}.$$

The following lemma shows that modulation space norms of compactly supported or bandlimited functions can be estimated using $\mathcal{F}L^p$ and $L^p$ norms respectively [22, 6, 8, 29].

**Proposition 6.** *For $K \subset \mathbb{R}^d$ compact and $p, q \in [1, \infty]$, there are constants $A, B, C, D > 0$ with*

*(i)* $A\|f\|_{\mathcal{F}L^q} \leq \|f\|_{M^{pq}} \leq B\|f\|_{\mathcal{F}L^q}, \quad f \in \mathcal{S}'(\mathbb{R}^d)$ *with* $supp\, f \subseteq K$*;*

*(ii)* $C\|f\|_{L^p} \leq \|f\|_{M^{pq}} \leq D\|f\|_{L^p}, \quad f \in \mathcal{S}'(\mathbb{R}^d)$ *with* $supp\, \widehat{f} \subseteq K.$

In the following, we shall denote norm equivalences as in statement $(i)$ above by

$$\|f\|_{\mathcal{F}L^q} \asymp \|f\|_{M^{pq}}, \quad f \in \mathcal{S}'(\mathbb{R}^d), \quad \operatorname{supp} f \subseteq K.$$

Similarly, statement $(ii)$ becomes

$$\|f\|_{L^p} \asymp \|f\|_{M^{pq}}, \quad f \in \mathcal{S}'(\mathbb{R}^d), \quad \operatorname{supp} \widehat{f} \subseteq K.$$

The symplectic Fourier transform of $F \in \mathcal{S}(\mathbb{R}^{2d})$ given in (1.1) is a $2d$-dimensional Fourier transform followed by a rotation of phase space by $\frac{\pi}{2}$. This implies that the symplectic Fourier transform shares most properties with the Fourier transform, for example, Proposition 6 remains true when replacing the Fourier transform by the symplectic Fourier transform.

Let $p_1, p_2, q_1, q_2 \in [1, \infty]$ and let $w$ be a $v$–moderate weight function on $\mathbb{R}^{4d}$. The weighted modulation space over phase space $\widetilde{M}_w^{p_1p_2q_1q_2}(\mathbb{R}^{2d})$ is the set of all tempered distributions $F \in \mathcal{S}'(\mathbb{R}^{2d})$ for which $\widetilde{V}_\psi F \in L_w^{p_1p_2q_1q_2}(\mathbb{R}^{4d})$.

Recapitulate that for $F \in \mathcal{S}'(\mathbb{R}^{2d})$, we have $\widetilde{V}_\psi F(x, t, \xi, \nu) = V_\psi F(x, \xi, \nu, -t)$,

$$\begin{aligned}\|F\|_{\widetilde{M}^{p_1p_2q_1q_2}} &= \|\widetilde{V}_\phi F\|_{L^{p_1p_2q_1q_2}} \\ &= \Big(\int\Big(\int\Big(\int\Big(\int |\widetilde{V}_\psi F(x,t,\xi,\nu)|^{p_1}\,dx\Big)^{p_2/p_1} dt\Big)^{q_1/p_2} d\xi\Big)^{q_2/q_1} d\nu\Big)^{1/q_1},\end{aligned}$$

and

$$\begin{aligned}\|F\|_{M^{p_1q_1q_2p_2}} &= \|V_\phi F\|_{L^{p_1q_1q_2p_2}} \\ &= \Big(\int\Big(\int\Big(\int\Big(\int |V_\psi F(x,\xi,\nu,t)|^{p_1}\,dx\Big)^{q_1/p_1} d\xi\Big)^{q_2/q_1} d\nu\Big)^{p_1/q_1} dt\Big)^{1/p_1},\end{aligned}$$

with usual adjustments if $p_1 = \infty$, $p_2 = \infty$, $q_1 = \infty$, and/or $q_2 = \infty$. This shows that the definition of $\widetilde{M}^{p_1,p_2,q_1,q_4}(\mathbb{R}^{2d})$ is based on changing the order of integration and on relabeling the integration exponents accordingly. Mixed $L^p$ spaces are sensitive towards the order of integration, and, hence $\widetilde{M}^{p_1p_2q_1q_2}(\mathbb{R}^{2d}) \nsubseteq M^{p_1p_2q_1q_2}(\mathbb{R}^{2d})$ and $M^{p_1p_2q_1q_2}(\mathbb{R}^{2d}) \nsubseteq \widetilde{M}^{p_1p_2q_1q_2}(\mathbb{R}^{2d})$ in general. But for $1 \le p \le q \le \infty$, Minkowski's inequality

$$\Big(\int\Big(\int |F(x,y)|^p\,dx\Big)^{q/p}dy\Big)^p \le \Big(\int\Big(\int |F(x,y)|^q\,dy\Big)^{p/q}dx\Big)^q$$

(with adjustments for $p=\infty$ and/or $q=\infty$ holds and implies the following.

**Proposition 7.** *Let $p_1, p_2, q_1, q_2 \in [1,\infty]$ and $w$ be a moderate weight function on $\mathbb{R}^{4d}$.*

*(a) If $p_2 \le \min\{q_1, q_2\}$, then $M_w^{p_1q_1q_2p_2}(\mathbb{R}^{2d}) \subseteq \widetilde{M}_w^{p_1p_2q_1q_2}(\mathbb{R}^{2d})$ and*
$\|\sigma\|_{\widetilde{M}_w^{p_1p_2q_1q_2}} \le \|\sigma\|_{M_w^{p_1q_1q_2p_2}}$.

*(b) If $\max\{q_1, q_2\} \le p_2$, then $\widetilde{M}_w^{p_1p_2q_1q_2}(\mathbb{R}^{2d}) \subseteq M_w^{p_1q_1q_2p_2}(\mathbb{R}^{2d})$ and*
$\|\sigma\|_{M_w^{p_1q_1q_2p_2}} \le \|\sigma\|_{\widetilde{M}_w^{p_1p_2q_1q_2}}$.

Note that results similar to ours could also be achieved using symbols in $M_w^{p_3p_4q_3q_4}(\mathbb{R}^{2d})$, but the so obtained results would be weaker and they would necessitate the additional condition $p_4 \le \min\{q_3, q_4\}$ .

The modulation space over phase space$\widetilde{M}_w^{p_1p_2q_1q_2}(\mathbb{R}^{2d})$ shares most of the properties of ordinary modulation spaces. For example, if $p_1 \le \widetilde{p}_1$, $p_2 \le \widetilde{p}_2$, $q_1 \le \widetilde{q}_1$ and $q_2 \le \widetilde{q}_2$, then

$$\widetilde{M}_w^{p_1p_2q_1q_2}(\mathbb{R}^{2d}) \subseteq \widetilde{M}_w^{\widetilde{p}_1\widetilde{p}_2\widetilde{q}_1\widetilde{q}_2}(\mathbb{R}^{2d}), \tag{2.3}$$

and

$$\|\sigma\|_{\widetilde{M}_w^{\widetilde{p}_1,\widetilde{p}_2,\widetilde{q}_1,\widetilde{q}_2}} \le \|\sigma\|_{\widetilde{M}_w^{p_1p_2q_1q_2}}, \quad \sigma \in \widetilde{M}_w^{p_1,p_2,q_2,q_2}(\mathbb{R}^{2d}).$$

Furthermore, let $p_1, p_2, q_1, q_2 \in [1,\infty]$. Then the dual of $\widetilde{M}_w^{p_1p_2q_1q_2}(\mathbb{R}^{2d})$ is $\widetilde{M}_w^{p_1'p_2'q_1'q_2'}(\mathbb{R}^{2d})$ where $p_1'$, $p_2'$, $q_1'$, $q_2'$ are conjugate exponenets of $p_1, p_2, q_1, q_2$ respectively.

The proofs of these results for modulation spaces over phase space are similar to the ones for the ordinary modulation spaces [14], and are omitted.

# 3. Comparison of Theorem 3 to results in the literature

Cordero and Nicola as well as Toft proved the following theorem on $M^{pq}$–boundedness for the class of pseudo-differential operators with symbols in $M^{s_1s_1s_2s_2}(\mathbb{R}^{2d})$, see Theorem 5.2 in [5] and Theorem 4.3 in [27].

**Theorem 8.** *Let $p, q, s_1, s_2 \in [1,\infty]$. Then for some $C > 0$,*

$$\|T_\sigma\|_{\mathcal{L}(M^{pq},M^{pq})} \le C\,\|\sigma\|_{M^{s_1,s_1,s_2,s_2}}, \quad \sigma \in M^{s_1,s_1,s_2,s_2}(\mathbb{R}^{2d}), \tag{3.1}$$

*if and only if*

$$s_2 \le \min\{p, p', q, q', s_1'\}.$$

Roughly speaking, to apply Theorem 8, we need to ensure that $\sigma(x,\xi)$ has $L^s$ 'decay' in $x$ and $\xi$ and that $\mathcal{F}\sigma(\nu,-t)=\mathcal{F}_s\sigma(t,\nu)$ has $L^{\min\{p,p',q,q',s'\}}$ 'decay' in $t$ and $\nu$. To apply Theorem 3, it suffices to ensure that $\sigma(x,\xi)$ has $L^{s_1}$ 'decay' in $x$ and $L^{s_2}$ 'decay'$\xi$, and that $\mathcal{F}_s\sigma(t,\nu)$ has $L^{\min\{p,p',s_1'\}}$ 'decay' in $t$ and $L^{\min\{q,q',s_2\}}$ 'decay' in $\nu$.

Using embeddings such as (2.3), we observe that indeed Theorem 8 provides boundedness of $T_\sigma$ if and only if

$$\sigma \in \bigcup_{s=\max\{p,p',q,q'\}}^{\infty} M^{s,s,s',s'} \subseteq \bigcup_{s=\max\{p,p',q,q'\}}^{\infty} \widetilde{M}^{s,s',s,s'} \tag{3.2}$$

while Theorem 3 provides boundedness of $T_\sigma$ if and only if

$$\sigma \in \bigcup_{s_1=\max\{p,p'\}}^{\infty} \bigcup_{s_2=\max\{q,q'\}}^{\infty} \widetilde{M}^{s_1,s_1',s_2,s_2'}.$$

To obtain the set inclusion in (3.2), we used Theorem 7 and the fact that $s\geq \max\{p,p'\}$ implies $s\geq 2\geq s'$.

As $L^2=M^{2,2}$, Theorem 8 implies the following $L^2$–boundedness result.

**Corollary 9.** *Let $r,s\in[1,\infty]$. Then for some $C>0$,*

$$\|T_\sigma\|_{\mathcal{L}(L^2,L^2)} \leq C\,\|\sigma\|_{M^{r,r,s,s}}, \quad \sigma\in M^{r,r,s,s}(\mathbb{R}^{2d}),$$

*if and only if*

$$s\leq \min\{2,r'\}.$$

Corollary 9 has been obtained earlier in 2003 by Gröchenig and Heil [15]. As comparison, we formulate the respective consequence of Theorem 3.

**Corollary 10.** *For $r,s\in[2,\infty]$, there exists a constant $C>0$ such that*

$$\|T_\sigma\|_{\mathcal{L}(L^2,L^2)} \leq C\,\|\sigma\|_{\widetilde{M}^{r,r',s,s'}}, \quad \sigma\in \widetilde{M}^{r,r',s,s'}(\mathbb{R}^{2d}).$$

As example, note that Theorem 8 does not imply that $T_\sigma: L^2(\mathbb{R}^d)\to L^2(\mathbb{R}^d)$ is bounded for $\sigma\in M^{\infty,2,2,1}(\mathbb{R}^{2d})$. But as $M^{\infty,2,2,1}(\mathbb{R}^{2d})\subseteq \widetilde{M}^{\infty,1,2,2}(\mathbb{R}^{2d})$, Theorem 3 indeed implies boundedness of $T_\sigma$ in this case.

For compositions of product and convolution operators, Theorem 3 implies the following result.

**Corollary 11.** *For $p,q\in[2,\infty]$, let $h_1\in M^{p,q'}(\mathbb{R}^d)$ and $h_2\in M^{p',q}(\mathbb{R}^d)$. Define*

$$Tf=h_1\cdot(h_2 * f), \quad f\in L^2(\mathbb{R}^d),$$

*and*

$$Hf=(h_1\cdot f) * h_2, \quad f\in L^2(\mathbb{R}).$$

*Then $T$ and $H$ are bounded operators on $L^2$ and moreover, there exist positive constants $C$ and $C'$ such that*

$$\|T\|_{\mathcal{L}(L^2,L^2)} \leq C\, \|h_1\|_{M^{p,q'}} \|h_2\|_{M^{p',q}},$$

*and*

$$\|H\|_{\mathcal{L}(L^2,L^2)} \leq C'\, \|h_1\|_{M^{p,q'}} \|h_2\|_{M^{p',q}}.$$

The proof of Corollary 11 follows immediately from Corollary 10, Lemma 20 and Lemma 21. Note that not separately, the convolution and multiplication operators above may not be bounded operators.

# 4. Proof of Theorem 3, Corollary 4, and Theorem 12

## 4.1. Proof of Theorem 12 and thereby of (1.5) and (1.6) implies (1.4) in Theorem 3

In this section we prove the weighted version of one implication of Theorem 3, that is the following theorem.

**Theorem 12.** *Let $w_1, w_2$ be moderate weight functions on $\mathbb{R}^{2d}$ and $w$ be a moderate weight function on $\mathbb{R}^{4d}$ that satisfies*

$$w(x,t,\nu,\xi) \leq w_1(x-t,\xi) w_2(x,\nu+\xi). \tag{4.1}$$

*Let $p_1, p_2, p_3, p_4, q_1, q_2, q_3, q_4 \in [1,\infty]$ be such that*

$$\begin{array}{l} \frac{1}{p_1'} + \frac{1}{p_2} \leq \frac{1}{p_3} + \frac{1}{p_4}, \quad p_4 \leq \min\{p_1', p_2\}, \\ \frac{1}{q_1'} + \frac{1}{q_2} \leq \frac{1}{q_3} + \frac{1}{q_4}, \quad q_4 \leq \min\{q_1', q_2\}. \end{array}$$

*Then there exists a constant $C > 0$ such that*

$$\|T_\sigma\|_{\mathcal{L}(M^{p_1 q_1}_{w_1}, M^{p_2 q_2}_{w_2})} \leq C\, \|\sigma\|_{\widetilde{M}^{p_3 p_4 q_3 q_4}_w}, \quad \sigma \in \widetilde{M}^{p_3 p_4 q_3 q_4}_w(\mathbb{R}^{2d}).$$

To prove Theorem 12 we need some preparation. For functions $f$ and $g$ in $\mathcal{S}(\mathbb{R}^d)$, the Rihaczek transform $R(f,g)$ of $f$ and $g$ is defined by

$$R(f,g)(x,\xi) = e^{2\pi i x\cdot\xi} \hat{f}(\xi) \overline{g(x)}.$$

For $\sigma \in \mathcal{S}(\mathbb{R}^{2d})$, pseudo-differential operators are related to Rihaczek transforms by

$$\big(T_\sigma f, g\big) = \big(\sigma, \overline{R(f,g)}\big)$$

for all functions $f$ and $g$ in $\mathcal{S}(\mathbb{R}^d)$. We define $A$, $T_A$ by

$$(T_A F)(x,t) = F(A(x,t)) = F(x-t,x)\,.$$

Then

$$\overline{R(f,g)}(x,\xi) = \mathcal{F}_{t\to\xi}\big(T_A(\overline{f}\otimes g)(x,\cdot)\big),$$

where

$$\mathcal{F}_{t\to\xi} f(\cdot + x) = \int e^{-2\pi i t\xi} f(t+x)\, dt.$$

**Lemma 13.** *Let $\varphi$ be a real valued Schwartz function on $\mathbb{R}^d$. Then for all $f$ and $g$ in $\mathcal{S}(\mathbb{R}^d)$*

$$V_{T_A(\varphi\otimes\varphi)} T_A(\overline{f}\otimes g)\,(x,t,\nu,\xi) = \overline{V_\varphi f(x-t,\xi)}\, V_\varphi g(x,\nu+\xi).$$

*Proof.* We compute

$$\begin{aligned}
&V_{T_A(\varphi\otimes\varphi)} T_A(\overline{f}\otimes g)\,(x,t,\nu,\xi)\\
&= \iint e^{-2\pi i(\widetilde{x}\nu+\widetilde{t}\xi)} T_A(\overline{f}\otimes g)(\widetilde{x},\widetilde{t}) T_A(\varphi\otimes\varphi)(\widetilde{x}-x,\widetilde{t}-t)\, d\widetilde{x}\, d\widetilde{t}\\
&= \int \Big( \int e^{-2\pi i \widetilde{t}\xi} \overline{f}(\widetilde{x}-\widetilde{t}) \varphi(\widetilde{x}-x-\widetilde{t}+t)\, d\widetilde{t} \Big) e^{-2\pi i \widetilde{x}\nu} g(\widetilde{x})\varphi(\widetilde{x}-x)\, d\widetilde{x}\\
&= \iint \overline{f}(s) g(\widetilde{x}) e^{-2\pi i \nu\widetilde{x} - 2\pi i \xi(\widetilde{x}-s)} \varphi(s-(x-t))\varphi(\widetilde{x}-x)\, d\widetilde{x}\, ds\\
&= \Big(\overline{\int e^{-2\pi i \xi s} f(s)\varphi(s-(x-t))\, ds}\Big)\Big(\int e^{-2\pi i(\nu+\xi)\widetilde{x}} g(\widetilde{x})\varphi(\widetilde{x}-x)\, d\widetilde{x}\Big)\\
&= \overline{V_\varphi f(x-t,\xi)}\; V_\varphi g(x,\nu+\xi).
\end{aligned}$$

□

**Lemma 14.** *Let $\varphi \in \mathcal{S}(\mathbb{R}^d)$ be a nonzero even real valued Schwartz function on $\mathbb{R}^d$. Then for all $f$ and $g$ in $\mathcal{S}(\mathbb{R}^d)$*

$$V_{\overline{R(\varphi,\varphi)}} \overline{R(f,g)}\,(x,\xi,\nu,t) = e^{-2\pi i \xi t}\; V_{T_A(\varphi\otimes\varphi)} T_A(\overline{f}\otimes g)\,(x,-t,\nu,\xi)\,.$$

*Proof.* For all $f$ and $g$ in $\mathcal{S}(\mathbb{R}^d)$

$$\begin{aligned}
&V_{\overline{R(\varphi,\varphi)}} \overline{R(f,g)}(x,\xi,\nu,t)\\
&= \iint e^{-2\pi i(\nu\widetilde{x}+t\widetilde{\xi})} \overline{R(f,g)(\widetilde{x},\widetilde{\xi})} R(\varphi,\varphi)(\widetilde{x}-x,\widetilde{\xi}-\xi)\, d\widetilde{x}\, d\widetilde{\xi}\\
&= \iint e^{-2\pi i(\nu\widetilde{x}+t\widetilde{\xi})} \mathcal{F}_{\widetilde{t}\to\widetilde{\xi}}\big(\overline{f}(\widetilde{x}-\cdot)\big) g(\widetilde{x}) \overline{\mathcal{F}_{\widetilde{t}\to\widetilde{\xi}-\xi}\big(\varphi(\widetilde{x}-x-\cdot)\big)} \varphi(\widetilde{x}-x)\, d\widetilde{x}\, d\widetilde{\xi}\\
&= \iint e^{-2\pi i(\nu\widetilde{x}+t\widetilde{\xi})} \mathcal{F}_{\widetilde{t}\to\widetilde{\xi}}\big(\overline{f}(\widetilde{x}-\cdot)\big) g(\widetilde{x}) \mathcal{F}_{\widetilde{t}\to\xi-\widetilde{\xi}}\big(\varphi(\widetilde{x}-x-\cdot)\big) \varphi(\widetilde{x}-x)\, d\widetilde{x}\, d\widetilde{\xi}.
\end{aligned} \tag{4.2}$$

On the other hand, Parseval's identity gives

$$\begin{aligned}
&V_{T_A(\varphi\otimes\varphi)}T_A(\overline{f}\otimes g)(x,t,\nu,\xi)\\
&= \iint e^{-2\pi i(\widetilde{x}\nu+\widetilde{t}\xi)}T_A(\overline{f}\otimes g)(\widetilde{x},\widetilde{t})T_A(\varphi\otimes\varphi)(\widetilde{x}-x,\widetilde{t}-t)\,d\widetilde{x}\,d\widetilde{t}\\
&= \int \Big(\int e^{-2\pi i\widetilde{t}\xi}\overline{f}(\widetilde{x}-\widetilde{t})\varphi(\widetilde{x}-x-\widetilde{t}+t)\,d\widetilde{t}\Big)e^{-2\pi i\widetilde{x}\nu}g(\widetilde{x})\varphi(\widetilde{x}-x)\,d\widetilde{x}\\
&= \iint \mathcal{F}_{\widetilde{t}\to\widetilde{\xi}}\big(\overline{f}(\widetilde{x}-\cdot)\big)\mathcal{F}^{-1}_{\widetilde{t}\to\widetilde{\xi}}\big(e^{-2\pi i\widetilde{t}\xi}\varphi(\widetilde{x}-x+t-\cdot)\big)e^{-2\pi i\widetilde{x}\nu}g(\widetilde{x})\varphi(\widetilde{x}-x)\,d\widetilde{\xi}\,d\widetilde{x}.
\end{aligned}$$

But,

$$\mathcal{F}^{-1}_{\widetilde{t}\to\widetilde{\xi}}\big(e^{-2\pi i\widetilde{t}\xi}\varphi(\widetilde{x}-x+t-\cdot)\big)=e^{-2\pi it(\xi-\widetilde{\xi})}\mathcal{F}_{\gamma\to\xi-\widetilde{\xi}}\big(\varphi(\widetilde{x}-x-\cdot)\big),$$

therefore,

$$\begin{aligned}
V_{T_A(\varphi\otimes\varphi)}T_A(\overline{f}\otimes g)(x,t,\nu,\xi) \;=\; & e^{-2\pi it\xi}\iint e^{2\pi i(t\widetilde{\xi}-v\widetilde{x})}\mathcal{F}_{\widetilde{t}\to\widetilde{\xi}}\big(\overline{f}(\widetilde{x}-\cdot)\big)\;\cdot\\
& \mathcal{F}_{\widetilde{t}\to\xi-\widetilde{\xi}}\big(\varphi(\widetilde{x}-x-\cdot)\big)g(\widetilde{x})\varphi(\widetilde{x}-x)\,d\widetilde{x}\,d\widetilde{\xi}.
\end{aligned}$$

Combining this identity with (4.2) completes the proof. □

**Proposition 15.** *Let $w_1, w_2, w$ be moderate functions that satisfy*

$$w(x,t,\nu,\xi)\le w_1(x-t,\xi)w_2(x,\nu+\xi).$$

*Let $\varphi$ be a nonzero real valued Schwartz function on $\mathbb{R}^d$ and define*

$$\mathcal{V}_{T_A(\varphi\otimes\varphi)}T_A(\overline{f}\otimes g)\,(x,t,\xi,\nu)=V_{T_A(\varphi\otimes\varphi)}T_A(\overline{f}\otimes g)\,(x,t,\nu,\xi) \tag{4.3}$$

*for all $f,g\in\mathcal{S}(\mathbb{R}^d)$ and $x,t,\xi,\nu\in\mathbb{R}^d$. If $p_1,p_2,p_3,p_4,q_1,q_2,q_3,q_4\in[1,\infty]$ satisfy*

$$\begin{array}{ll}\frac{1}{p_1}+\frac{1}{p_2}=\frac{1}{p_3}+\frac{1}{p_4}, & p_3\le\min\{p_1,p_2,p_4\},\\ \frac{1}{q_1}+\frac{1}{q_2}=\frac{1}{q_3}+\frac{1}{q_4}, & q_3\le\min\{q_1,q_2,q_4\},\end{array} \tag{4.4}$$

*then*

$$\|\mathcal{V}_{T_A(\varphi\otimes\varphi)}T_A(\overline{f}\otimes g)\|_{L^{p_3p_4q_3q_4}_w}\le\|f\|_{M^{p_1q_1}_{w_1}}\|g\|_{M^{p_2\cdot q_2}_{w_2}}.$$

*Proof.* By Lemma 13, we have

$$\mathcal{V}_{T_A(\varphi\otimes\varphi)}T_A(\overline{f}\otimes g)\,(x,t,\xi,\nu)=\overline{V_\varphi f(x-t,\xi)}\,V_\varphi g(x,\nu+\xi).$$

So, by (4.1), for $t,\xi,\nu\in\mathbb{R}^d$,

$$\begin{aligned}
&\|w(\cdot,t,\xi,\nu)\mathcal{V}_{T_A(\varphi\otimes\varphi)}T_A(\overline{f}\otimes g)(\cdot,t,\xi,\nu)\|_{L^{p_3}}\\
&\le \Big(\int|w_1(x-t,\xi)\Big(V_\varphi f\Big)(x-t,\xi)|^{p_3}|w_2(x,\nu+\xi)\Big(V_\varphi g\Big)(x,\nu+\xi)|^{p_3}\,dx\Big)^{1/p_3}\\
&= \Big(|w_2(\cdot,\nu+\xi)V_\varphi g(\cdot,\nu+\xi)|^{p_3}*|w_1(\cdot,\xi)V_\varphi f(\cdot,\xi)|^{p_3}\,(t)\Big)^{1/p_3}.
\end{aligned}$$

Then, (4.4) implies

$$\frac{1}{r_1} + \frac{1}{s_1} = 1 + \frac{1}{a_1},$$

with $r_1 = p_2/p_3 \geq 1$, $s_1 = p_1/p_3 \geq 1$ and $a_1 = p_4/p_3 \geq 1$, hence, we can apply Young's inequality and obtain

$$\begin{aligned}
&\|w(\cdot,\cdot,\xi,\nu)\mathcal{V}_{T_A(\varphi\otimes\varphi)}T_A(\overline{f}\otimes g)(\cdot,\cdot,\xi,\nu)\|_{L^{p_3,p_4}} \\
&= \; \||w_2(\cdot,\nu+\xi)V_\varphi g(\cdot,\nu+\xi)|^{p_3} * |w_1(\cdot,\xi)V_\varphi f(\cdot,\xi)|^{p_3}\|_{L^{a_1}}^{1/p_3} \\
&\leq \; \||w_2(\cdot,\nu+\xi)V_\varphi g(\cdot,\nu+\xi)|^{p_3}\|_{L^{r_1}}^{1/p_3} \; \||w_1(\cdot,\xi)V_\varphi f(\cdot,\xi)|^{p_3}\|_{L^{s_1}}^{1/p_3}.
\end{aligned} \tag{4.5}$$

To estimate (4.5) further, we note that integrating with respect to $\xi$ can be again considered a convolution. In fact (4.4) leads to

$$\frac{1}{r_2} + \frac{1}{s_2} = 1 + \frac{1}{a_2},$$

where $r_2 = q_2/q_3$, $s_2 = q_1/q_3$ and $a_2 = q_4/q_3$. Young's inequality then implies

$$\begin{aligned}
&\|w\mathcal{V}_{T_A(\varphi\otimes\varphi)}T_A(\overline{f}\otimes g)\|_{L^{p_3p_4q_3q_4}} \\
&\leq \; \Big(\int\Big(\int|w_2(x,y)V_\varphi g(x,y)|^{p_3r_1}\,dx\Big)^{(r_2q_3)/(p_3r_1)}dy\Big)^{1/(r_2q_3)} \cdot \\
&\qquad \Big(\int\Big(\int|w_1(x,y)V_\varphi f(x,y)|^{p_3s_1}\,dx\Big)^{(s_2q_3)/(p_3s_1)}dy\Big)^{(1/s_2q_3)} \\
&= \; \|f\|_{M^{p_1q_1}_{w_1}}\|g\|_{M^{p_2q_2}_{w_2}},
\end{aligned}$$

which completes the proof. $\square$

Now, we are ready to give sufficient conditions on the boundedness of pseudo-differential operators with symbols in $\widetilde{M}^{p_3,p_4,q_3.q_4}(\mathbb{R}^{2d})$.

**Lemma 16.** *Let $w_1, w_2, w$ be moderate weight functions that satisfy (4.1). Let $p_1,p_2,p_3,p_4,q_1,q_2,q_3,q_4 \in [1,\infty]$ be such that*

$$\begin{aligned}
\tfrac{1}{p_3} &\in \left[\tfrac{1}{p_1'} + \tfrac{1}{p_2} - \tfrac{1}{p_4}, \min\{\tfrac{1}{p_1'}, \tfrac{1}{p_2}, \tfrac{1}{p_4}\}\right], \\
\tfrac{1}{q_3} &\in \left[\tfrac{1}{q_1'} + \tfrac{1}{q_2} - \tfrac{1}{q_4}, \min\{\tfrac{1}{q_1'}, \tfrac{1}{q_2}, \tfrac{1}{q_4}\}\right].
\end{aligned} \tag{4.6}$$

*Then there exists a constant $C > 0$ such that*

$$\|T_\sigma\|_{\mathcal{L}(M^{p_1q_1}_{w_1}, M^{p_2q_2}_{w_2})} \leq C\,\|\sigma\|_{\widetilde{M}^{p_3p_4q_3q_4}_w}, \quad \sigma \in \widetilde{M}^{p_3p_4q_3q_4}_w(\mathbb{R}^{2d}). \tag{4.7}$$

*Proof.* Let us first assume $p_1,p_2,p_3,p_4,q_1,q_2,q_3,q_4 \in [1,\infty]$ satisfy (4.6) and in addition

$$\frac{1}{p_1'} + \frac{1}{p_2} = \frac{1}{p_3} + \frac{1}{p_4} \quad \text{and} \quad \frac{1}{q_1'} + \frac{1}{q_2} = \frac{1}{q_3} + \frac{1}{q_4}. \tag{4.8}$$

Let $f,g\in\mathcal{S}(\mathbb{R}^d)$. Since the dual of $\widetilde{M}_w^{p_3p_4q_3q_4}(\mathbb{R}^{2d})$ is $\widetilde{M}_w^{p_3',p_4',q_3',q_4'}(\mathbb{R}^{2d})$, it follows that

$$\begin{aligned}|(T_\sigma f,g)| &= |(\sigma,\overline{R(f,g)})|\\ &\le \|\sigma\|_{\widetilde{M}_w^{p_3p_4q_3q_4}}\|\overline{R(f,g)}\|_{\widetilde{M}_w^{p_3',p_4',q_3',q_4'}}.\end{aligned}$$

To obtain (4.7), it is enough to show that there exists $C>0$ such that

$$\|\overline{R(f,g)}\|_{\widetilde{M}_w^{p_3',p_4',q_3',q_4'}}\le C\,\|f\|_{M_{w_1}^{p_1q_1}}\|g\|_{M_{w_2}^{p_2',q_2'}}.$$

Let $\varphi$ be a nonzero real valued even function in $\mathcal{S}(\mathbb{R}^d)$. Then by Lemma 14,

$$\begin{aligned}\left|\mathcal{V}_{\overline{R(\varphi,\varphi)}}\overline{R(f,g)}\,(x,t,\xi,\nu)\right| &= \left|V_{\overline{R(\varphi,\varphi)}}\overline{R(f,g)}\,(x,\xi,\nu,-t)\right|\\ &= \left|V_{T_A(\varphi\otimes\varphi)}T_A(\overline{f}\otimes g)\,(x,t,\nu,\xi)\right|\\ &= \left|\mathcal{V}_{T_A(\varphi\otimes\varphi)}T_A(\overline{f}\otimes g)\,(x,t,\xi,\nu)\right|.\end{aligned}$$

where $\mathcal{V}_{T_A(\varphi\otimes\varphi)}$ is defined in (4.3). Therefore, by Proposition 15, we have

$$\begin{aligned}\|R(f,g)\|_{\widetilde{M}_w^{p_3',p_4',q_3',q_4'}} &= \|\mathcal{V}_{T_A(\varphi\otimes\varphi)}T_A(\overline{f}\otimes g)\|_{L_w^{p_3',p_4',q_3',q_4'}}\\ &\le \|f\|_{M_{w_1}^{p_1q_1}}\|g\|_{M_{w_2}^{p_2',q_2'}}.\end{aligned}$$

To obtain (4.7) in the general case, that is $p_1,p_2,p_3,p_4,q_1,q_2,q_3,q_4\in[1,\infty]$ satisfy (4.6) but not necessarily (4.8), set

$$\frac{1}{\widetilde{p}_2}=\frac{1}{p_3}+\frac{1}{p_4}-\frac{1}{p_1'}\quad\text{and}\quad\frac{1}{\widetilde{q}_2}=\frac{1}{q_3}+\frac{1}{q_4}-\frac{1}{q_1'}.$$

Then it is easy to see that $\widetilde{p}_2\le p_2$, $\widetilde{q}_2\le q_2$ and $p_1,\widetilde{p}_2,p_3,p_4,q_1,\widetilde{q}_2,q_3,q_4\in[1,\infty]$ satisfy (4.6). Hence

$$\|T_\sigma f\|_{M_{w_2}^{p_2q_2}}\le C\,\|T_\sigma f\|_{M_{w_2}^{\widetilde{p}_2,\widetilde{q}_2}}\le\|f\|_{M_{w_1}^{p_1q_1}}\|\sigma\|_{\widetilde{M}_w^{p_3p_4q_3q_4}},$$

for some $C>0$. □

**Proof of Theorem 12:** Let $f\in\mathcal{S}(\mathbb{R}^d)$. Set

$$\frac{1}{\widetilde{p}_3}=\frac{1}{p_1'}+\frac{1}{p_2}-\frac{1}{p_4}\quad\text{and}\quad\frac{1}{\widetilde{q}_3}=\frac{1}{q_1'}+\frac{1}{q_2}-\frac{1}{q_4}.$$

Then it is easy to see that

$$\widetilde{p}_3\ge p_3,\quad \widetilde{q}_3\ge q_3.$$

Furthermore, $\{p_1,p_2,\widetilde{p}_3,p_4,q_1,q_2,\widetilde{q}_3,q_4\}$ satisfies (4.6), therefore there exist $C_1,C_2>0$ such that

$$\begin{aligned}\|T_\sigma f\|_{M_{w_2}^{p_2q_2}} &\le C_1\|f\|_{M_{w_1}^{p_1q_1}}\|\sigma\|_{\widetilde{M}_w^{\widetilde{p}_3,p_4,\widetilde{q}_3,q_4}}\\ &\le C_2\|f\|_{M_{w_1}^{p_1q_1}}\|\sigma\|_{\widetilde{M}_w^{p_3p_4q_3q_4}}.\end{aligned}$$

## 4.2. Proof of Corollary 4

Let $1 \leq p, q \leq 2$. By Theorem 3, $T_\sigma : M^{p,p'} \to M^q$ is bounded. Using the bounded embeddings $M^p \subset L^p \subset M^{p,p'}$ for all $1 \leq p \leq 2$ (for more details see [8]), it follows that $T_\sigma : L^p(\mathbb{R}^d) \to L^q(\mathbb{R}^d)$ is bounded. Similarly, using $M^{p,p'} \subset L^p \subset M^p$ for all $q \geq 2$, we can prove $T_\sigma : L^p(\mathbb{R}^d) \to L^q(\mathbb{R}^d)$ is bounded for $p, p_3, p_4, q, q_3, q_4$ satisfying (b) or (c) or (d) in Corollary 4. $\square$

## 4.3. Proof of (1.4) implies (1.5), (1.6) in Theorem 3

To show necessity of (1.5) and (1.6) in Theorem 3, we shall use two mixed $L^p$ norms on phase space, namely,

$$\|F\|_{L^{pq}} = \Big( \int \Big( \int |F(x,\xi)|^p \, dx \Big)^{q/p} d\xi \Big)^{1/q},$$

and

$$\|F\|_{\widetilde{L}^{pq}} = \Big( \int \Big( \int |F(x,\xi)|^q \, d\xi \Big)^{p/q} dx \Big)^{1/p},$$

for $p, q \in [1, \infty)$. For $p = \infty$ and/or $q = \infty$ we make the usual adjustment.

Similarly, we can define $\widetilde{M}^{pq}(\mathbb{R}^d)$ to be the space of all functions $f \in \mathcal{S}'(\mathbb{R}^d)$ for which

$$\|f\|_{\widetilde{M}^{pq}} = \|V_\varphi f\|_{\widetilde{L}^{pq}} < \infty,$$

where $\varphi \in \mathcal{S}(\mathbb{R}^d) \setminus \{0\}$. Note that it can be easily checked that

$$\|f\|_{\widetilde{M}^{pq}} = \|\widehat{f}\|_{M^{qp}}.$$

Below, we use an idea from the proof of Proposition 6 given in [22] to prove the following lemma.

**Lemma 17.** *Let $K \subset \mathbb{R}^{2d}$ be compact. Then*

$$\|\sigma\|_{\widetilde{M}^{p_3 p_4 q_3 q_4}} \asymp \|\sigma\|_{\mathcal{F}\widetilde{L}^{q_4 p_4}}, \quad \sigma \in \mathcal{S}'(\mathbb{R}^{2d}), \quad \operatorname{supp} \sigma \subset K.$$

*Proof.* Choose $r > 0$ with $\operatorname{supp} \sigma \subseteq B_r^{2d}(0)$, where

$$B_r^{2d}(0) = \{x \in \mathbb{R}^{2d} : \|x\| \leq r\}$$

is the Euclidean unit ball in $\mathbb{R}^{2d}$ with center 0, radius $r$ and Lebesgue measure $|B_r^{2d}(0)|$. Let $\psi \in \mathcal{S}(\mathbb{R}^{2d})$ with $supp\, \psi \subset B_r^{2d}(0)$. Then it is easy to see that

$$\big|\widetilde{V}_\psi \sigma\big|(x,t,\xi,\nu) = \big|\sigma * M_{\nu,-t}\overline{\widetilde{\psi}}\big|(x,\xi),$$

where

$$\widetilde{\psi}(x,\xi) = \psi(-x,-\xi).$$

Therefore, for fixed $t, \nu$ we have

$$\begin{aligned} \operatorname{supp} \Big( \Big|\widetilde{V}_\psi \sigma\Big| (\cdot, t, \cdot, \nu) \Big) &\subseteq \operatorname{supp}(\sigma) + supp\, (M_{\nu,-t}\overline{\widetilde{\psi}}) \\ &\subseteq B_r^{2d}(0) + B_r^{2d}(0) \subseteq B_{2r}^{2d}(0). \end{aligned} \tag{4.9}$$

Let $\xi \in B_{2r}^d(0)$. Then by (4.9),

$$\begin{aligned}
\|\widetilde{V}_\psi \sigma(\cdot,t,\xi,\nu)\|_{L^{p_3}(\mathbb{R}^d)}^{p_3} &= \int_{B_{2r}^d(0)} \left|\widetilde{V}_\psi \sigma\right|^{p_3}(x,t,\xi,\nu)\,dx \\
\leq\ & |B_{2r}^d(0)|\,\left\|\widetilde{V}_\psi \sigma(\cdot,t,\xi,\nu)\right\|_{L^\infty} = |B_{2r}^d(0)|\,\left\|\sigma * M_{\nu,-t}\overline{\widetilde{\psi}}(\cdot,\xi)\right\|_{L^\infty} \\
\leq\ & |B_{2r}^d(0)|\,\left\|\sigma * M_{\nu,-t}\overline{\widetilde{\psi}}\right\|_{L^\infty} \leq |B_{2r}^d(0)|\,\left\|\mathcal{F}^{-1}\big(\widehat{\sigma}\,T_{\nu,-t}\overline{\widehat{\widetilde{\psi}}}\big)\right\|_{L^\infty} \\
\leq\ & |B_{2r}^d(0)|\,\left\|\widehat{\sigma}T_{\nu,-t}\overline{\widehat{\widetilde{\psi}}}\right\|_{L^1} = |B_{2r}^d(0)|\,\big(|\widehat{\sigma}| * |\widehat{\psi}|\big)(-\nu,t)
\end{aligned} \tag{4.10}$$

On the other hand, if $\xi \in \mathbb{R}^d \setminus B_{2r}^d(0)$, then by (4.9),

$$\|\widetilde{V}_\psi \sigma(\cdot,t,\xi,\nu)\|_{L^{p_3}} = 0. \tag{4.11}$$

Therefore, (4.10) and (4.11) imply

$$\begin{aligned}
&\|\sigma\|_{\widetilde{M}^{p_3p_4q_3q_4}} = \left\|\widetilde{V}_\psi \sigma\right\|_{L^{p_3p_4q_3q_4}} \\
\leq\ & |B_{2r}^d(0)|^{1/p_3} \int \Big( \int_{B_{2r}^d(0)} \Big( \int \Big( (|\widehat{\sigma}| * |\widehat{\psi}|(-\nu,t))^{p_4}\,dt\Big)^{q_3/p_4} d\xi\Big)^{q_4/q_3} d\nu\Big)^{1/q_4} \\
\leq\ & |B_{2r}^d(0)|^{(1/p_3)+(1/q_3)} \Big( \int \Big( \int (|\widehat{\sigma}| * |\widehat{\psi}|(-\nu,t))^{p_4}\,dt\Big)^{q_4/p_4} d\nu\Big)^{1/q_4} \\
\leq\ & |B_{2r}^d(0)|^{(1/p_3)+(1/q_3)} \left\| |\widehat{\sigma}| * |\widehat{\psi}| \right\|_{\widetilde{L}^{q_4,p_4}} \\
\leq\ & |B_{2r}^d(0)|^{(1/p_3)+(1/q_3)} \|\widehat{\sigma}\|_{\widetilde{L}^{q_4,p_4}} \|\widehat{\psi}\|_{\widetilde{L}^{1,1}} \leq C\|\widehat{\sigma}\|_{\widetilde{L}^{q_4,p_4}}.
\end{aligned}$$

Now, let $\psi \in C^\infty(\mathbb{R}^{2d})$ be compactly supported with $\psi \equiv 1$ on $B_{2r}^{2d}(0)$. Let $\chi_{B_r^{2d}(0)}$ be the characteristic function on $B_r^{2d}(0)$. Then using $supp\ \sigma \subseteq B_r^{2d}(0)$, it follows that for all $x,t,\xi,\nu \in \mathbb{R}^d$,

$$\begin{aligned}
&\chi_{B_{2r}^{2d}(0)}(x,\xi)\,\widetilde{V}_\psi \sigma(x,t,\xi,\nu) \\
=\ & \chi_{B_{2r}^{2d}(0)}(x,\xi) \int_{B_r^{2d}(0)} \sigma(\widetilde{x},\widetilde{\xi})\,e^{-2\pi i(\widetilde{x}\nu - \widetilde{\xi}t)}\,\psi(\widetilde{x}-x,\widetilde{\xi}-\xi)\,d\widetilde{x}\,d\widetilde{\xi} \\
=\ & \chi_{B_{2r}^{2d}(0)}(x,\xi) \int_{B_r^{2d}(0)} \sigma(\widetilde{x},\widetilde{\xi})\,e^{-2\pi i(\widetilde{x}\nu - \widetilde{\xi}t)}\,d\widetilde{x}\,d\widetilde{\xi} \\
=\ & \chi_{B_{2r}^{2d}(0)}(x,\xi)\mathcal{F}\sigma(\nu,-t).
\end{aligned}$$

Hence,

$$\begin{aligned}
&\|\sigma\|_{\widetilde{M}^{p_3p_4q_3q_4}} = \left\|\widetilde{V}_\psi \sigma\right\|_{L^{p_3p_4q_3q_4}} \geq \left\|\chi_{B_{2r}^{2d}(0)}\,\widetilde{V}_\psi \sigma\right\|_{L^{p_3p_4q_3q_4}} \\
=\ & \Big( \int \Big( \int \Big( \int \Big( \int \left|\chi_{B_{2r}^{2d}(0)}(x,\xi)\,\mathcal{F}\sigma(\nu,-t)\right|^{p_3} dx\Big)^{p_4/p_3} dt\Big)^{q_3/p_4} d\xi\Big)^{q_4/q_3} d\nu\Big)^{1/q_4} \\
=\ & \left\|\chi_{B_{2r}^{2d}(0)}\right\|_{L^{p_3q_3}}\,\|\sigma\|_{\mathcal{F}\widetilde{L}^{q_4,p_4}}
\end{aligned}$$

which completes the proof. $\square$

**Lemma 18.** *Let $\lambda > 0$ and $\varphi_\lambda(x) = e^{-\pi\lambda|x|^2}$. Then for $\lambda \geq 1$,*

$$\|\varphi_\lambda\|_{M^{pq}} \asymp \|\varphi_\lambda\|_{\widetilde{M}^{pq}} \asymp \lambda^{-d/q'},$$

*and*

$$\|\varphi_{\lambda^{-1}}\|_{M^{pq}} \asymp \|\varphi_{\lambda^{-1}}\|_{\widetilde{M}^{pq}} \asymp \lambda^{d/p}.$$

The proof of Lemma 18 is an immediate corollary of Lemma 3.2 in [4] and is omitted here.

**Lemma 19.** *Let $K \subset \mathbb{R}^d$ be compact. For $h \in C^\infty(\mathbb{R}^d)$ and $\lambda \geq 1$ set $h_\lambda(x) = h(x)e^{-\pi i\lambda|x|^2}$. Then for all $p, q \in [1, \infty]$,*

$$\|h_\lambda\|_{M^{pq}} \asymp \|\widehat{h}_\lambda\|_{L^q} \asymp \lambda^{d/q-d/2}, \quad h \in C^\infty(\mathbb{R}^d), \quad \operatorname{supp} h \subset K.$$

Lemma 19 is well known and its proof can be found in, for example, [5].

**Lemma 20.** *Let $h_1, h_2 \in \mathcal{S}(\mathbb{R}^d)$ and*

$$\eta(t,\nu) = e^{-2\pi it\nu} h_2(t)\widehat{h}_1(\nu), \quad t, \nu \in \mathbb{R}^d.$$

*If $\sigma = \widetilde{\mathcal{F}}\eta$. Then we have*

$$\sigma(x,\xi) = (M_\xi h_2 * h_1)(x) \tag{4.12}$$

*and*

$$T_\sigma f = (h_1 f) * h_2, \quad f \in \mathcal{S}(\mathbb{R}^d). \tag{4.13}$$

*Moreover,*

$$\|\sigma\|_{\widetilde{M}^{p_3p_4q_3q_4}} = \|h_1\|_{M^{p_3,q_4}} \|h_2\|_{M^{p_4,q_3}}.$$

*Proof.* Clearly, (4.12) and (4.13) hold. Now, let $\varphi$ be any nonzero real valued Schwartz function on $\mathbb{R}^d$. Let

$$\psi(t,\nu) = \varphi(t)\varphi(\nu)e^{-2\pi it\nu}.$$

and define

$$\widetilde{\psi}(x,\xi) = \widetilde{\mathcal{F}}\psi(-x,-\xi).$$

Then

$$\begin{aligned} \left|\widetilde{V}_{\widetilde{\psi}}\sigma(x,t,\xi,\nu)\right| &= \left|\left(\sigma, M_{\nu.-t}T_{x,\xi}\widetilde{\psi}\right)\right| \\ &= \left|\left(\widetilde{\mathcal{F}}\eta, \widetilde{\mathcal{F}}(T_{-t,\nu}M_{-\xi,x}\psi)\right)\right|. \end{aligned}$$

Now since $\widetilde{\mathcal{F}}$ is a unitary operator, it follows that

$$
\begin{aligned}
\left|\left(\widetilde{V}_{\widetilde{\psi}}\sigma\right)(x,t,\xi,\nu)\right| &= \left|\left(\eta, T_{-t,\nu}M_{-\xi,x}\psi\right)\right| \\
&= \left|\iint \eta(\widetilde{t},\widetilde{\nu})\, e^{2\pi i\xi(\widetilde{t}+t)}\, e^{-2\pi i x(v-\widetilde{\nu})}\overline{\psi}(t+\widetilde{t},\widetilde{\nu}-\nu)\, d\widetilde{t}\, d\widetilde{\nu}\right| \\
&= \left|\iint \widehat{h}_1(\widetilde{\nu})h_2(\widetilde{t})\varphi(\widetilde{\nu}-\nu)\varphi(\widetilde{t}+t)\, e^{-2\pi i\widetilde{\nu}(x-t)}\, e^{2\pi i\widetilde{t}(\nu-\xi)}\, d\widetilde{t}\, d\widetilde{\nu}\right| \\
&= |(V_\varphi\widehat{h}_1)(\nu, x-t)|\, |(V_\varphi h_2)(-t,\nu-\xi)|.
\end{aligned}
$$

Hence,

$$\|\sigma\|_{\widetilde{M}^{p_3p_4q_3q_4}} = \|h_1\|_{M^{p_3,q_4}}\|h_2\|_{M^{p_4,q_3}}.$$

$\square$

Similarly, we can prove the following.

**Lemma 21.** *Let Let $h_1, h_2 \in \mathcal{S}(\mathbb{R}^d)$ and $\sigma = h_1 \otimes \widehat{h}_2$. Then*

$$T_\sigma f = h_1 \cdot (h_2 * f), \quad f \in \mathcal{S}(\mathbb{R}^d)$$

*and*

$$\|h_1 \otimes \widehat{h}_2\|_{\widetilde{M}^{p_3p_4q_3q_4}} = \|h_1\|_{M^{p_3,q_4}}\|\widehat{h}_2\|_{\widetilde{M}^{q_3,p_4}}.$$

**Proof of (1.4) implies (1.5) and (1.6) in Theorem 3:** Let $h \in C^\infty(\mathbb{R}^d)$ be chosen with compact support and $h(0) = 1$ and $h(x) \geq 0$ for all $x \in \mathbb{R}^d$. Then for any $\lambda \geq 1$, we define $h_\lambda$ and $\sigma_\lambda$ respectively by

$$h_\lambda(x) = h(x)e^{-\pi i\lambda|x|^2}.$$

and

$$\sigma_\lambda(x,\xi) = h\otimes h_\lambda(x,\xi) = h(x)h_\lambda(\xi).$$

Let $f_\lambda = \mathcal{F}^{-1}\overline{h_\lambda}$. Then $f_\lambda \in \mathcal{S}(\mathbb{R}^d)$ and

$$T_{\sigma_\lambda}f_\lambda(x) = \int e^{2\pi i x\xi}h(x)|h(\xi)|^2\, d\xi.$$

So, $T_{\sigma_\lambda}f_\lambda$ is independent of $\lambda$. Since $\sigma_\lambda$ has compact support, by Lemma 17 and Lemma 19

$$
\begin{aligned}
\|\sigma_\lambda\|_{\widetilde{M}^{p_3p_4q_3q_4}} &\asymp \|\mathcal{F}\sigma_\lambda\|_{\widetilde{L}^{q_4,p_4}} \\
&= \|\widehat{h}\|_{L^{q_4}(\mathbb{R}^d)}\|\widehat{h}_\lambda\|_{L^{p_4}(\mathbb{R}^d)} \\
&\asymp \lambda^{(d/p_4)-(d/2)}.
\end{aligned} \tag{4.14}
$$

Moreover, by Lemma 6 and Lemma 19, since $\mathcal{F}f_\lambda$ has compact support,

$$\|f_\lambda\|_{M^{p_1q_1}(\mathbb{R}^d)} = \|f_\lambda\|_{L^{p_1}(\mathbb{R}^d)} \asymp \lambda^{(d/p_1)-(d/2)}. \tag{4.15}$$

Hence by (1.4), (4.14) and (4.15), there exists $C > 0$ such that for all $\lambda \geq 1$

$$\|T_\sigma f_\lambda\|_{M^{p_2 q_2}(\mathbb{R}^d)} \leq C\, \lambda^{(d/p_4)+(d/p_1)-d}.$$

But $\|T_\sigma f_\lambda\|_{M^{p_2 q_2}(\mathbb{R}^d)}$ is nonzero and independent of $\lambda$, therefore $\frac{d}{p_4} + \frac{d}{p_1} - d \geq 0$, and $p_4 \leq p_1'$.
To prove $q_4 \leq q_1'$, we let $h_1 = \overline{f} = h_\lambda$ and $h_2 \in \mathcal{S}(\mathbb{R}^d)$ be such that $\widehat{h}_2$ is compactly supported, independent of $\lambda$ and

$$\|(h_1 f) * h_2\|_{L^{p_2}(\mathbb{R}^d)} \neq 0.$$

Let $\sigma = \widetilde{\mathcal{F}}\eta$ where

$$\eta(t,\nu) = \widehat{h}_1(\nu) h_2(t) e^{-2\pi i t\nu}.$$

Then by Lemma 20 and (1.4)

$$\|(h_1 f) * h_2\|_{L^{p_2}(\mathbb{R}^d)} \leq C\, \|\widehat{h}_1\|_{L^{q_4}(\mathbb{R}^d)} \|h_2\|_{L^{p_4}(\mathbb{R}^d)} \|\widehat{f}\|_{L^{q_1}(\mathbb{R}^d)},$$

for some constant $C > 0$. So, by Lemma 19 for all $\lambda \geq 1$

$$\|(h_1 f) * h_2\|_{L^{p_2}(\mathbb{R}^d)} \leq C\, \lambda^{(d/q_4)-(d/2)} \lambda^{(d/q_1)-(d/2)},$$

but $\|(h_1 f) * h_2\|_{L^{p_2}(\mathbb{R}^d)}$ is nonzero and independent of $\lambda$, therefore $(d/q_4) + (d/q_1) - d \geq 0$ and, hence, $q_4 \leq q_1'$.

Now, let $h_1 = f = \varphi_\lambda$ and $h_2 = \varphi_{\lambda^{-1}}$, where $\varphi_\lambda$ and $\varphi_{\lambda^{-1}}$ are defined in Lemma 18. If we let $\sigma = h_1 \otimes h_2$. Then by Lemma 18 and Lemma 21, for $\lambda \geq 1$ we have

$$\|\sigma\|_{\widetilde{M}^{p_3 p_4 q_3 q_4}(\mathbb{R}^{2d})} \asymp \lambda^{d/q_3 - d/q_4'}$$

and $\|f\|_{M^{p_1 q_1}(\mathbb{R}^d)} \asymp \lambda^{-d/q_1'}$. On the other hand $T_\sigma f$ is also a Gaussian function and it can be easily checked that

$$\|T_\sigma f\|_{M^{p_2 q_2}(\mathbb{R}^d)} \asymp \lambda^{-d/q_2'}.$$

Therefore by (1.4)

$$\lambda^{d/q_3 - d/q_4' - d/q_1' + d/q_2'} \geq 1$$

for all $\lambda \geq 1$. Hence, we get

$$\frac{1}{q_1'} + \frac{1}{q_2} \leq \frac{1}{q_3} + \frac{1}{q_4}.$$

Similarly, by letting $h_1 = f = \varphi_{\lambda^{-1}}$ and $h_2 = \varphi_\lambda$, we get

$$\frac{1}{p_1'} + \frac{1}{p_2} \leq \frac{1}{p_3} + \frac{1}{p_4}.$$

Again assume $\sigma$ has the form given in Lemma 20. Let $h(x) = f(x) = e^{-\pi|x|^2/2}$ and $h_2 = \varphi_{\lambda^{-1}}$. Then $T_\sigma$ is also a Gaussian function, moreover by Lemma 18 and (1.4) for all $\lambda \geq 1$

$$\lambda^{d/p_4 - d/p_2} \geq C,$$

for some $C > 0$. Hence $p_4 \leq p_2$.
To prove $q_4 \leq q_2$, we let

$$\sigma(x,\xi) = e^{2\pi i x\xi} h_1(x) h_2(\xi),$$

where $h_1$ and $h_2$ are compactly supported Schwartz functions on $\mathbb{R}^d$. Then $\sigma$ is compactly supported and therefore by Lemma 17,

$$\|\sigma\|_{\widetilde{M}^{p_3p_4q_3q_4}(\mathbb{R}^{2d})} = \|\mathcal{F}\sigma\|_{L^{p_4,q_4}(\mathbb{R}^{2d})}.$$

On the other hand, by an easy calculation, we have

$$\big|\mathcal{F}\sigma\big|(\nu,t) = \big|V_{h_1}\widehat{h}_2\big|(t,\nu) = \big|V_{\widehat{\overline{h_2}}}\overline{h_1}\big|(t,\nu).$$

Therefore,

$$\|\sigma\|_{\widetilde{M}^{p_3p_4q_3q_4}(\mathbb{R}^{2d})} \leq C_{h_2}\, \|\widehat{h}_1\|_{L^{q_4}(\mathbb{R}^d)}, \tag{4.16}$$

and

$$\|\sigma\|_{\widetilde{M}^{p_3p_4q_3q_4}(\mathbb{R}^{2d})} \leq C_{h_1}\, \|\widehat{h}_2\|_{L^{p_4}(\mathbb{R}^d)},$$

where $C_{h_1}$ and $C_{h_2}$ are positive constants depending on $h_1$ and $h_2$ respectively. Let $h_1 = h_\lambda$ and $h_2$ be any compactly supported function and $f$ be a Schwartz function on $\mathbb{R}^d$ and both $h_2$ and $\widehat{f}$ be independent of $\lambda$ such that $(h_2, \overline{\widehat{f}}) \neq 0$. Then

$$\begin{aligned} \|T_\sigma f\|_{M^{p_2q_2}(\mathbb{R}^d)} &= \|h_1\|_{M^{p_2q_2}(\mathbb{R}^d)} |(h_2, \overline{\widehat{f}})| \\ &= |(h_2, \overline{\widehat{f}})| \|\widehat{h}_1\|_{L^{q_2}(\mathbb{R}^d)} \asymp \lambda^{(d/q_2)-(d/2)}, \end{aligned} \tag{4.17}$$

and by (4.16)

$$\|\sigma\|_{\widetilde{M}^{p_3p_4q_3q_4}(\mathbb{R}^{2d})} \leq C_{h_2}\, \lambda^{(d/q_4)-(d/2)}.$$

Hence, (4.17) and (1.4) imply

$$\lambda^{(d/q_4)-(d/q_2)} \geq C,$$

where $C > 0$ is independent of $\lambda \geq 1$. Hence $(d/q_4) - (d/q_2) \geq 0$ which implies that $q_4 \leq q_2$. $\square$

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